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On the Relation between Maximum Modulus, Maximum Term, and Taylor Coefficients of an Entire Function

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Relations are established between the quantities in the title in form of direct estimates, instead of measuring the growth of f by its order, type, or some generalized order. For entire functions of relatively slow growth, a distinct increase of precision is achieved. Our approach originates in work by Hadamard, Le Roy, Valiron, and Berg.

1. INTRODUCTION

Let f be an entire function with Taylor expansion

$$f(z)=\sum_{n=0}^{\infty} a_n z^n, \tag{1}$$

maximum modulus

$$M(r)=M(r,f)=\max_{|z|\leqslant r} |f(z)| \qquad (r\geqslant 0), \tag{2}$$

and maximum term

$$m(r)=m(r,f)=\max_{n\in \mathbb{P}} |a_n r^n| \qquad (r\geqslant 0), \tag{3}$$

where $\mathscr{P}=\{0,1,2,...\}$. One way of characterizing the growth of an entire function in terms of its Taylor coefficients is to relate the a_n with order and type of f. Thus, supposing $0<\rho<\infty$ and $0\leqslant\tau<\infty$, one has $\limsup_{n\to\infty} n\,|a_n|^{\rho/n}=\tau\rho e$ if and only if f is of order ρ and type τ. Probably Pringsheim

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[18] was the first to formulate it in this way; see also Valiron [29] and Boas [4].

Apparently, this characterization is rather coarse, and it is limited to functions with order $\rho > 0$ and type $\tau > 0$, thus to functions with $M(r)$ growing like $\rho\sqrt{\tau}\, r^{\rho/2} e^{\tau r^\rho}$ at least. In recent years, there has been an increasing interest in refinements and extensions of the concept of order. These were mainly designed for functions with $\rho = \infty$, thus for rapidly increasing functions (see, e.g., [26, 22, 21, 23, 19, 25, 11, 1, 16]), but some of them also apply to more slowly increasing functions, for which the classical order is zero (e.g., [26, 22, 19, 11, 12, 17]). These papers work with various kinds of iterated or generalized orders, defined as limits of the form

$$\limsup_{r \to \infty} \frac{\alpha(\log M(r))}{\beta(\log r)}$$

for certain functions α and β. Using such concepts of an order, somewhat finer relations between the growth and the Taylor coefficients of an entire function can be established, but, as compared with a direct $\mathcal{O}$-estimate of $|a_n|$ or $M(r)$, these are still rather crude.

As for the relation between the Taylor coefficients and the maximum term $m(r)$, or that between $M(r)$ and $m(r)$, the situation is similar. For example, relations of the form

$$\limsup_{r \to \infty} \frac{\alpha(\log M(r))}{\beta(\log r)} = \limsup_{r \to \infty} \frac{\alpha(\log m(r))}{\beta(\log r)}$$

for certain functions α, β have been established in [24, 29, 26, 25, 16, 12].

The purpose of the present paper is to set up more precise interrelations between $M(r)$, $m(r)$, and a_n, for entire functions of relatively slow growth, in terms of direct estimates for these quantities. In this respect we refer to the older papers on the subject, e.g., to those by Hadamard [9] in 1893, Le Roy [14] in 1900, Valiron [28] in 1914, and also to the theory of asymptotic expansions which was influenced by them, and to papers by Wiman [30] in 1914, Bakhoom [2] in 1933, and Hayman [10] in 1956. In these papers direct estimates of the desired precision were studied, and it even was attempted to derive asymptotic relations for $f(z)$ itself, instead of $M(r)$, out of the growth of the a_n. In the latter respect, the more slowly increasing functions turned out to have disappointing properties, cf., e.g., Le Roy [14, Sect. 6, p. 264], Valiron [28, p. 260], and Mattson [15]. But when confining attention to $M(r)$ or $m(r)$, it will turn out that just for the slowly increasing functions the most precise interrelations between growth, Taylor coefficients, and further characteristics hold (see Theorems 1 and 3 and Corollary 2). Here we call an entire function slowly increasing if $M(r)$ increases, essentially, not faster than

$$\exp\left\{c(\alpha-1)\left[\frac{\log 2r}{c\alpha}\right]^{\alpha/(\alpha-1)}\right\} \tag{4}$$

for $\alpha=2$ and an arbitrary $c>0$. (The critical value $\alpha=2$ has been found to be significant also in connection with a conjecture of Erdös.) But also for more rapidly increasing functions there are direct estimates of $M(r)$, $m(r)$, and the a_n. If f increases like (4) with an $\alpha\in(1,2)$, for example, Proposition 2 and Corollary 1 apply, and, for still more rapidly increasing f, including functions of arbitrary classical order $\rho>0$, Proposition 1 and Theorem 2 apply. Though in the latter case a necessary and sufficient characterization of the growth of f in terms of the a_n is impossible, the results are still sharper than the limit relations mentioned above.

2. Maximum Modulus and Taylor Coefficients

We first prove two propositions for a class of entire functions which includes all f of order ρ for some $\rho>0$. Then we restrict ourselves to entire functions of slow growth in order to obtain a characterization theorem of the desired precision.

We denote by $C^2[x_1,\infty)$ the class of twice continuously differentiable functions on $[x_1,\infty)$ and set, for any $g\in C^2[x_1,\infty)$ with $g''(x)>0$,

$$A(r)=\exp\{(g')^{-1}(\log r)\log r-g((g')^{-1}(\log r))\}. \tag{5}$$

The following proposition follows from Cauchy's inequality $|a_n|\leqslant r^{-n}M(r)$ by substituting $r=\frac{1}{2}\exp g'(n)$.

Proposition 1. *Let f be an entire function with Taylor series* (1) *and maximum modulus* (2), *and let $g\in C^2[x_1,\infty)$ be such that $g'(x)\to\infty$ as $x\to\infty$ and $g''(x)>0$ for each $x\geqslant x_1$. If*

$$M(r)=\mathcal{O}(A(2r)),\qquad r\to\infty,$$

it follows that

$$|a_n|=\mathcal{O}(2^n/\varphi(n)),\qquad n\to\infty, \tag{6}$$

where $\varphi(x)=\exp g(x)$, $x\geqslant x_1$.

This estimate is contained, e.g., in [10, p. 68], where it has the form $|a_n|\leqslant M(r_n)(r_n)^{-n}$. It is also contained in [6, p. 183], in the form $|a_n|\leqslant$

$(r(n))^{-n}\exp(h(r(n)))$, provided that $M(r)=\mathcal{O}(e^{h(r)})$, $r\to\infty$, and where $r(x)$ is the inverse function of $xh'(x)$. This turns into (6) by substituting $h(x)=\log A(2x)$.

We now turn to the reverse implication. Following Berg [3, p. 270], we define Γ to be the class of functions $g\in C^2[x_1,\infty)$ for some $x_1\geqslant 0$ for which there exists a function ω such that

$$x-\omega(x)\geqslant x_1, \qquad x\geqslant x_1,$$

$$\lim_{x\to\infty} g'(x)=\infty, \qquad \lim_{x\to\infty} \omega^2(x)\, g''(x)=\infty, \qquad \lim_{x\to\infty} g''(x)=0,$$

$$g''(x+\theta\omega(x))\sim g''(x) \qquad \text{as } x\to\infty, \text{ uniformly in } \theta \text{ for } |\theta|\leqslant 1.$$

Here "$\sim$" means that $\lim_{x\to\infty} g''(x+\theta\omega(x))/g''(x)=1$.

PROPOSITION 2. *Let f be an entire function with Taylor series* (1) *satisfying* (6) *for some* φ *such that* $g(x)=\log\varphi(x)\in\Gamma$. *Then*

$$M(r)=\mathcal{O}([g''((g')^{-1}(\log 2r))]^{-1/2}A(2r)), \qquad r\to\infty. \tag{7}$$

Proof. By definition one has

$$M(r)\leqslant\sum_{n=0}^{\infty}|a_n|\,r^n, \qquad r>0,$$

and inserting (6) here, it follows that

$$M(r)=\mathcal{O}\left(\sum_{n=0}^{\infty}\frac{(2r)^n}{\varphi(n)}\right), \qquad r\to\infty.$$

Since $g\in\Gamma$, the asymptotic representation

$$\sum_{n=0}^{\infty}\frac{(2r)^n}{\varphi(n)}\sim\sqrt{2\pi}\,[g''((g')^{-1}(\log 2r))]^{-1/2}A(2r), \qquad r\to\infty, \tag{8}$$

given by Berg [3, Theorem 28.3] implies the assertion.

Remarks. (a) The conclusions of Propositions 1 and 2 are best possible in the sense that the $\mathcal{O}$ cannot be replaced by o in (6) and (7), respectively. In case φ grows at least as rapidly as $\exp(x^\tau)$, $\tau\geqslant 2$, this will be a consequence of Theorem 1 below. For Proposition 2 and general φ with $g\in\Gamma$ this is also clear from (8), by choosing f with $a_n=2^n/\varphi(n)$.

(b) In connection with Proposition 2 the paper [10] by Hayman needs to be mentioned. There an asymptotic relation for the a_n in terms of $M(r)$ is established (see [10, Theorem I and Corollaries]) which generalizes Stirling's formula. In general one cannot compare the results of

[10] with Proposition 2 because our hypothesis (6) does not imply that f is an "admissible" function in the sense of [10]. But in the particular case of $f(z)=e^z$, where the result of [10] reduces to Stirling's formula, it can be concluded that (7) is sharp. Indeed, setting $g(n)=n(\log 2+\log n)+\frac{1}{2}\log n-n$, (6) is satisfied and, without using $(g')^{-1}$ explicitly, it can be seen that $M(r)$ is asymptotically equal to the right-hand side of (7) then. To verify that $g\in\Gamma$ one may choose $\omega(x)=x^{3/4}$ in this case.

(c) Precursors of Berg, not mentioned in [3], are Le Roy [14] in 1900 and Valiron [28] in 1914, who formulated roughly the result of Proposition 2 (see [14, Sect. 5, pp. 261–262], replacing there α, $\varphi(x)$, ξ by $\log r$, $g(x)-x\log 2$, $(g')^{-1}(\log 2r)$, respectively, and [28, p. 260]).

Now we turn to the case of slowly increasing entire functions. Expressed in terms of a particular $\varphi(x)$ of the form $\varphi(x)=\exp(cx^\alpha)$, $c>0$, this means that the following theorem will cover the cases $\alpha\geqslant 2$, whereas Propositions 1 and 2 cover the cases $\alpha>1$ and $1<\alpha<2$, respectively. Let $\tilde{\Gamma}$ denote the set of functions $g\in C^2[x_1,\infty)$ for some $x_1\geqslant 0$, with the properties that

$$g''(x)>0 \text{ for each } x\geqslant x_1\text{, and either } \lim_{x\to\infty} g''(x)=\infty,$$

$$g'''(x) \text{ exists for } x\geqslant x_1 \text{ and } \lim_{x\to\infty} g'''(x)(g''(x))^{-3/2}=0,$$

$$\text{or } \lim_{x\to\infty} g''(x)=c \text{ for some } c>0.$$

THEOREM 1. *Let f be an entire function with Taylor series* (1), *let $\varphi(x)=e^{g(x)}$ for some $g\in\tilde{\Gamma}$, and let $A(r)$ be defined by* (5). *The following are equivalent*:

(i) $M(r)=\mathcal{O}(A(2r))$, $r\to\infty$,

(ii) $|a_n|=\mathcal{O}(2^n/\varphi(n))$, $n\to\infty$.

Proof. The implication (i) $\Rightarrow$ (ii) is contained in Proposition 1. For the converse, consider first the case when $\lim_{x\to\infty} g''(x)=c>0$ with $c<\infty$. Then the proof follows as in Proposition 2, using the second part of [3, Theorem 28.3]. If $\lim_{x\to\infty} g''(x)=\infty$, it follows again that $M(r)=\mathcal{O}(\sum_{n=0}^{\infty}(2r)^n/\varphi(n))$, $r\to\infty$, and we have to show that the latter sum is $\mathcal{O}(A(2r))$ as $r\to\infty$. To this end we set

$$h(x,t)=xt-g(t)$$

and use a result of Sirovich [27, pp. 96–98] (see also Evgrafov [6, p. 18, (9)])

$$\int_0^\infty e^{h(x,t)}\,dt\sim e^{h(x,t_0(x))}\left\{\frac{2\pi}{-h_{tt}(x,t_0(x))}\right\}^{1/2},\qquad x\to\infty. \tag{9}$$

Here h_{tt} denotes the second derivative with respect to t, and $t_0(x) = (g')^{-1}(x)$. The hypotheses of [27, p. 98, case 2] are satisfied since, for each $x > x_1$, $h(x, t)$ as a function of t has a global maximum at $t_0 = t_0(x) = (g')^{-1}(x)$, i.e., $h_t(x, t_0(x)) = 0$. Moreover, $h_{tt}(x, t_0(x)) = -g''((g')^{-1}(x)) \neq 0$ and, observing that $(g')^{-1}(x) \to \infty$ as $x \to \infty$, it follows by the definition of $\tilde{\Gamma}$ that

$$-\lim_{x \to \infty} h_{tt}(x, t_0(x)) = \infty,$$

as well as

$$\lim_{x \to \infty} h_{ttt}(x, t_0(x))(h_{tt}(x, t_0(x)))^{-3/2} = 0.$$

Setting now $x_r = t_0(\log 2r)$ and $k_r = [x_r]$, where $[x_r]$ denotes the integral part of x_r, the sum in question can be majorized as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(2r)^n}{\varphi(n)} &= \sum_{n=0}^{\infty} \exp\{h(\log 2r, n)\} \\ &\leqslant \int_0^{k_r} \exp\{h(\log 2r, t)\}\, dt + \exp\{h(\log 2r, k_r)\} \\ &\quad + \exp\{h(\log 2r, k_r + 1)\} \\ &\quad + \int_{k_r + 1}^{\infty} \exp\{h(\log 2r, t)\}\, dt \\ &\leqslant 2 \exp\{h(\log 2r, x_r)\} + \int_0^{\infty} \exp\{h(\log 2r, t)\}\, dt.\end{aligned}$$

Thus, using (9), the definition of h, and (5), we have

$$M(r) = \mathcal{O}(A(2r)\{2 + \sqrt{2\pi}(g''((g')^{-1}(\log 2r)))^{-1/2}\}), \qquad r \to \infty,$$

and since $g''((g')^{-1}(\log 2r)) \to \infty$, as $r \to \infty$, assertion (i) follows. This completes the proof.

In connection with the particular case $\varphi(x) = \exp(cx^{\alpha})$, $c > 0$, of Theorem 1 we mention the papers by Bakhoom [2] and Le Roy [14]. Theorem 1 then states that, for each $\alpha \geqslant 2$,

$$M(r) = \mathcal{O}\left(\exp\left\{c(\alpha - 1)\left(\frac{\log 2r}{c\alpha}\right)^{\alpha/(\alpha - 1)}\right\}\right), \qquad r \to \infty,$$

holds if and only if

$$|a_n| = \mathcal{O}(2^n/\exp(cn^{\alpha})), \qquad n \to \infty.$$

In [2, 14] the asymptotic expansion (9) is given for this φ, in the cases $\alpha \in \mathbb{N}$ and $\alpha > 2$, respectively.

3. Maximum Term and Taylor Coefficients

A satisfactory characterization of $m(r)$ in terms of the Taylor coefficients holds for a large class of entire functions, including those of order $\rho > 0$ and type $\tau \geq 0$ (choose $\varphi(x) = \exp g(x)$, $g(x) = (x/\rho)\log(x2^\rho/\rho e \tau)$ below).

THEOREM 2. *Let f be an entire function with Taylor series* (1) *and maximum term* (3), *and let g, φ, $A(r)$ be given as in Proposition* 1. *Condition* (6) *is equivalent to*

$$m(r) = \mathcal{O}(A(2r)), \qquad r \to \infty. \tag{10}$$

Proof. Let (6) be satisfied, so that for each $r > 0$, $n \geq n_0$,

$$|a_n r^n| \leq M \frac{(2r)^n}{\varphi(n)},$$

where n_0, M are constants. For fixed r, the maximum over x of the function $(2r)^x/\varphi(x)$ is attained at $x = (g')^{-1}(\log 2r)$, and has value $A(2r)$, provided that $(g')^{-1}(\log 2r) > x_1$. Therefore

$$\max_{n \geq n_0} |a_n r^n| \leq M \max_{n \geq n_0} \frac{(2r)^n}{\varphi(n)} \leq MA(2r)$$

for each r larger than some r_0. Choosing $r_1 > r_0$ large enough, so that $|a_{n_0} r^{n_0}| \geq \max\{|a_n r^n|;\ 0 \leq n < n_0\}$ for each $r > r_1$, it follows that also $m(r) \leq MA(2r)$, $r > r_1$. Conversely, (10) implies that

$$|a_n| \leq M \frac{A(2r)}{r^n} \qquad (r > r_0), \tag{11}$$

where r_0, M are constants. Choosing $r = \frac{1}{2}e^{g'(n)}$ for some $n \in \mathbb{N}$ in (11), we have $r > r_0$ for n large enough (since $g'(x) \to \infty$ as $x \to \infty$), and (6) follows.

4. Maximum Modulus and Maximum Term

In view of Cauchy's inequality, the estimate $m(r) \leq M(r)$ is immediate. Estimates in the inverse direction have been studied by, among others, Valiron [29, pp. 32–34], who proved for entire functions f of order ρ the inequality $M(r) < m(r)\, r^{\rho+\varepsilon}$, $r > r_\varepsilon$, where $\varepsilon > 0$ is arbitrary, and by Wiman [30, pp. 306, 315], who proved e.g., that $M(r) < m(r)(\log m(r))^{1/2+\varepsilon}$ holds for infinitely many r, provided that f is an entire transcendental function and $\varepsilon > 0$, and that $M(r) < \rho_1 \sqrt{2\pi}\, m(r)(\log m(r))^{1/2}$, $r > r_0$, where $\rho_1 > \rho$, provided f is of order ρ. The latter inequality is a special case of Corollary 1 below (choosing $g(x) = (x/\rho) \log(x 2^\rho / \rho e \tau)$ there and using that then $m(r) = \mathcal{O}(e^{\tau r^\rho})$, $r \to \infty$, in view of Theorem 2). Indeed, both then imply that $M(r) = \mathcal{O}(\rho \sqrt{\tau}\, r^{\rho/2} e^{\tau r^\rho})$, $r \to \infty$, for an entire f of order $\rho > 0$ and type $\tau > 0$.

COROLLARY 1. *Let f be an entire function with Taylor series* (1), *satisfying $m(r) = \mathcal{O}(A(2r))$, $r \to \infty$, with $A(r)$ defined by* (5), *for some $\varphi(x) = \exp g(x)$, $g \in \Gamma$. Then*

$$M(r) = \mathcal{O}([g''((g')^{-1}(\log 2r))]^{-1/2} A(2r)), \qquad r \to \infty.$$

This follows by combining Theorem 2 with Proposition 2. The next corollary, which is a consequence of Theorems 2 and 1, is concerned with functions of slow growth only.

COROLLARY 2. *Let f be given as in Theorem* 1. *The following are equivalent*:

(i) $m(r) = \mathcal{O}(A(2r))$, $r \to \infty$,

(ii) $M(r) = \mathcal{O}(A(2r))$, $r \to \infty$.

For further relations between $m(r)$ and $M(r)$ compare Rosenbloom [20] and Kövari [13]. In connection with Corollary 2, it is also interesting to compare the papers of Clunie and Hayman [5] and Gray and Shah [8] which deal with a conjecture of Erdös. Denoting by $\alpha(f)$ and $\beta(f)$ the lim sup and lim inf of $m(r)/M(r)$, respectively, as $r \to \infty$, the conjecture was that there can occur only the two cases $\alpha(f) > \beta(f)$ or $\alpha(f) = \beta(f) = 0$. In [5, 8] this is confirmed for certain classes of functions, and disproved for others. In particular, it was shown in [8] that, given any Φ with $\lim_{r \to \infty} \Phi(r) = \infty$, the class of functions f with property

$$\lim_{r \to \infty} \frac{\log m(r)}{\Phi(r)(\log r)^2} = 0$$

always contains an f for which $\alpha(f) = \beta(f) = 0$ and

$$\lim_{r \to \infty} \frac{\log m(r)}{(\log r)^2} = \infty. \tag{12}$$

This implies that for such f an equivalence like Corollary 2 cannot hold, and hence that the hypotheses of Corollary 2 cannot be relaxed essentially. Indeed, choosing $\varphi(x)=\exp(cx^\alpha)$ there, our hypotheses exclude the case $\alpha<2$. In terms of the estimate $m(r)=\mathcal{O}(A(r))$ this means that we exclude functions f for which $m(r)$ increases faster than $\exp\{a(\log r)^2\}$ for some constant $a>0$. Condition (12) expresses just the same hypothesis.

5. Characterization Theorem for Slowly Increasing Functions

Summarizing the results for slowly increasing functions, and combining them with the characterizations in [7], we have the following theorem. Here $E_n[f, C[-1, 1]]$ denotes the error of best uniform approximation by algebraic polynomials on the interval $[-1, 1]$, $c_k(f)$ are the Fourier–Chebychev coefficients of f, and $A(r)$ is defined by (5).

Theorem 3. *Let f be an entire function with Taylor series* (1), *and let $\varphi(x)=e^{g(x)}$ for some $g\in\tilde{\Gamma}$. The following assertions are equivalent*:

(i) $E_n[f, C[-1,1]]=\mathcal{O}(1/\varphi(n+1))$, $n\to\infty$,

(ii) $\|f^{(r)}\|_{C[-1,1]}=\mathcal{O}(2^r r!/\varphi(r))$, $r\to\infty$,

(iii) $|f^{(r)}(0)|=\mathcal{O}(2^r r!/\varphi(r))$, $r\to\infty$,

(iv) $|c_k(f)|=\mathcal{O}(1/\varphi(k))$, $k\to\infty$,

(v) $M(r)=\mathcal{O}(A(2r))$, $r\to\infty$,

(vi) $m(r)=\mathcal{O}(A(2r))$, $r\to\infty$.

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